

# Analytic Function

محاضرة رياضية (٤)

٢٠١٥/٤/٢٣

الدالة القابلة للتحليل هي الدالة التي تُعرف قبل الاشتقاق أي تعرف قانون النهاية للمشتقة الأولى عند النقطة وجوارها أي عند النقطة والنقطة المجاورة لها وهذا المفهوم الحقيقي .

□ Differentiable at a point :- الاشتقاق عند نقطة

$$f'(z) = \lim_{\Delta z \rightarrow 0} (z + \Delta z) -$$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\lim_{x \rightarrow x_0} \lim_{y \rightarrow y_0} ( ) = \lim_{y \rightarrow y_0} \lim_{x \rightarrow x_0} ( )$$



Example. show that  $f(z) = \bar{z} = x - iy$  is not differentiable at  $z_0 = 0$

solution

$$f'(0) = \lim_{\Delta z \rightarrow 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z}$$

$$f(z) = \bar{z}, f(0) = 0$$

$$f(0 + \Delta z) = f(\Delta z) = \overline{\Delta z} = \overline{\Delta x + i\Delta y} = \Delta x - i\Delta y$$

$$f'(0) = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$\lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = 1$$

$$\lim_{\Delta y \rightarrow 0} \lim_{\Delta x \rightarrow 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{-i\Delta y}{i\Delta y} = -1$$

since  $\lim \lim \neq \lim \lim$

the limit doesn't exist.



## Remark

[1] الدالة تكون (diff) إذا تحقق تعريف النهاية عند النقطة.

[2] " " (Analytic) إذا تحقق تعريف النهاية عند النقطة وجوارها.

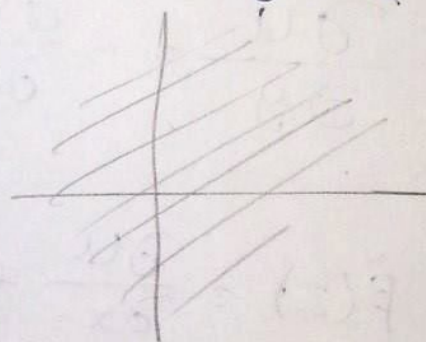
Ex  $f(z) = \frac{1}{z-1}$

↳ this  $f_n$  is Analytic in summing points at  $z=1$

[3] الدالة تسمى (entire) إذا كانت (Analytic) على  $z$ -plane بالكامل.

$e^z, \sin z, \cos z, \dots, z^n, \dots$

مثال



توضيح [4] الدالة التي تصوى على  $\bar{z}$  أو  $|z|$  يمكن أن تكون ~~ليست~~ <sup>توصف</sup> ~~ليست~~ <sup>ليست</sup>

ليست  $\Leftrightarrow$  diff  $\Leftrightarrow$  not Analytic  $\Leftrightarrow$  not entire



## \* Cauchy - Riemann equation

$$Z = x + iy$$

$$\text{or } Z = r e^{i\theta}$$

$$\text{I} \quad f(z) = u(x, y) - i v(x, y)$$

الدالة تكون (Analytic) إذا تحقق

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

$$\tilde{f}(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial x} - i \frac{\partial u}{\partial y}$$

$$= \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

بدلالة الدالة  $u \leftarrow$

لو عارفين الجزء الحقيقي فقط من الممكن أن نحل على المشتقة

الأولى ~~و~~ والعكس ~~بالعكس~~ صحيح.



$$\boxed{2} \quad f(z) = u(r, \theta) + iv(r, \theta) \longrightarrow (\text{polar form})$$

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

C.R (polar)

$$\tilde{f}(z) = e^{-i\theta} (u_r + i v_r)$$

$$= \frac{e^{-i\theta}}{r} [v_\theta - i u_\theta] = e^{-i\theta} \left[ u_r - \frac{i}{r} u_\theta \right]$$

$$= e^{-i\theta} \left[ \frac{1}{r} v_\theta - \frac{i}{r} u_r \right]$$



### Example

Discuss that the following fns are analytic or not.

1]  $f(z) = \cos z$

2]  $f(z) = \bar{z}$

الحنا، رفين الهالست  
Analytic

3]  $f(z) = z^2 + 5iz + 3 - i$

4]  $f(z) = z^3$

Sol

1]

$$f(z) = \cos z = \cos(x+iy)$$

$$= \cos x \cdot \cos iy - \sin x \sin iy$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$u = \cos x \cosh y$$

$$v = -\sin x \sinh y$$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y$$

$$\frac{\partial v}{\partial x} = -\cos x \sinh y$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y$$

$$\frac{\partial v}{\partial y} = -\sin x \cosh y$$

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$$u_x = v_y$$

$$, u_y = -v_x$$

\* the  $f_n$  is analytic

$$\boxed{2} \quad f(z) = \bar{e}^z$$

sol

$$f(z) = \bar{e}^z = e^{x-iy} = e^x e^{-iy}$$

$$e^{-iy} = \cos(-y) + i \sin(-y) = \cos y - i \sin y$$

$$f(z) = e^x (\cos y - i \sin y)$$

$$u = e^x \cos y$$

$$v = -e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y$$

$$\frac{\partial v}{\partial x} = -e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial y} = -e^x \cos y$$

$$u_x \neq v_y, \quad u_y \neq -v_x$$

→ The  $f_n$  is not Analytic.

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$$\boxed{3} \quad f(z) = z^3$$

$$f(z) = (r e^{i\theta})^3 = r^3 e^{i3\theta}$$

$$= r^3 [\cos(3\theta) + i \sin(3\theta)]$$

$$u = r^3 \cos(3\theta)$$

$$v = r^3 \sin(3\theta)$$

$$\frac{\partial u}{\partial r} = 3r^2 \cos(3\theta) \quad \frac{\partial v}{\partial r} = 3r^2 \sin(3\theta)$$

$$\frac{\partial u}{\partial \theta} = -3r^3 \sin(3\theta) \quad \frac{\partial v}{\partial \theta} = 3r^3 \cos(3\theta)$$

$$u_r = \frac{1}{r} v_\theta, \quad v_r = -\frac{1}{r} u_\theta$$

$\Rightarrow$  The function is analytic



Example 3 use C-R equations to show

that  $\frac{\partial z^n}{\partial z} = n z^{n-1}$

Solution

المسألة مطلوبة فيها أن نوجد قانون المشتقة الأولى (C-R) ونفصح أنها بحد الاختصار تكون  $n z^{n-1}$ .

$$f(z) = z^n = (r e^{i\theta})^n = r^n e^{in\theta}$$

$$= \underbrace{r^n \cos(n\theta)}_u + \underbrace{r^n \sin(n\theta)}_v$$

$$\tilde{f}(z) = e^{-i\theta} \left( \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \right)$$

$$= e^{-i\theta} \left[ n r^{n-1} \cos(n\theta) + i n r^{n-1} \sin(n\theta) \right]$$

$$= n r^{n-1} e^{-i\theta} \left[ \cos(n\theta) + i \sin(n\theta) \right]$$

□



$$e^{i*} = \cos * + i \sin *$$

$$\hat{f}(z) = n r^{n-1} e^{-i\theta} \left[ e^{in\theta} \right]$$

$$= n r^{n-1} e^{i(n-1)\theta} = n (r e^{i\theta})^{n-1}$$

$$\boxed{\hat{f}(z) = n z^{n-1}}$$

→ Deduce the form of C-R eqn for  
 $f(z) = u(x,y) + i v(x,y)$

Sol

$$f(z_0) = \lim_{\Delta x \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$\lim_{\Delta y \rightarrow 0} \lim_{\Delta x \rightarrow 0} = \lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0}$$

نعرف أن النهاية موجودة



$$f(z) = u(x, y) + i v(x, y)$$

$$f(z_0) = u(x_0, y_0) + i v(x_0, y_0)$$

$$f(z_0 + \Delta z)$$

Note  $z_0 + \Delta z = x_0 + i y_0 + (\Delta x + i \Delta y)$

$$= (x_0 + \Delta x) + i (y_0 + \Delta y)$$

$$f(z_0 + \Delta z) = u(x_0 + \Delta x, y_0 + \Delta y) + i v(x_0 + \Delta x, y_0 + \Delta y)$$

$$f'(z_0) = \lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i \Delta y} +$$

$$i \lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i \Delta y}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta y \rightarrow 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta y}$$

تغیر  $x$  ،  $y$  ثابت

□□□



$$\bar{f}(z_0) = \frac{\partial u}{\partial x} - i \frac{\partial v}{\partial x} \longrightarrow \textcircled{1}$$

since  $\lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} = \lim_{\Delta y \rightarrow 0} \lim_{\Delta x \rightarrow 0}$

$0 = \Delta x$  و  $0 = \Delta y$

$$\bar{f}(z_0) = \lim_{\Delta y \rightarrow 0} \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta x + i \Delta y}$$

$$+ i \lim_{\Delta y \rightarrow 0} \lim_{\Delta x \rightarrow 0} \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x + i \Delta y}$$

$$= \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i \Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i \Delta y}$$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \longrightarrow \textcircled{2}$$



From (1), (2)

$$u_x = -v_y, \quad v_x = u_y$$

$$u_x, v_y$$

### The Harmonic fn

→ الدالة  $u$  تسمى دالة توافقية إذا حققت معادلة Laplace

$$u_{xx} = u_{yy} = 0$$

In polar

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$$

→ نلاحظ أنه إذا كانت الدالة (Analytic) فإن الجزء

الحقيقي والتخيل وهو  $u$  يكون دوال توافقية لذلك يمكن

لإستخدام هذه المعلومة لتعيين قيمة ثوابت أو مجاهيل

في صورة  $u, v$ .